AWP15-ENR-01/JSI-02

EUROfusion

Fast Model Predictive Control for Magnetic Plasma Control

MPC using fast online 1st-order QP methods

Matija Perne

Jožef Stefan Institute

CREATE PPCC team

cosylab CONTROL SYSTEM LABORATORY

SLOVENIAN RESEARCH AGENCY

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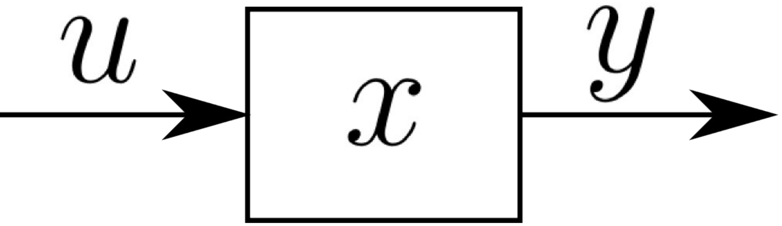
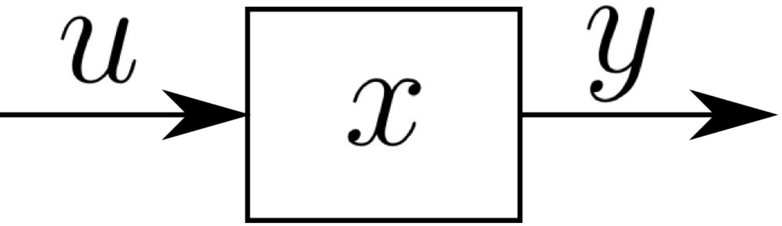
Linear dynamic system, optimum



- Discrete-time model:

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

Linear dynamic system, optimum



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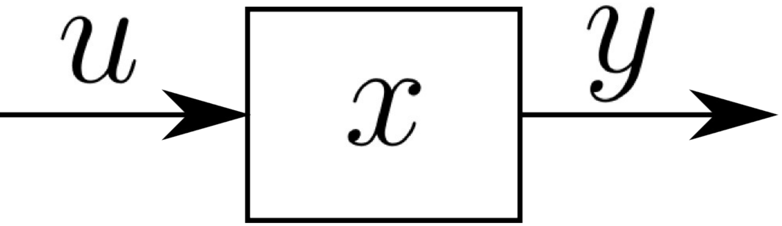


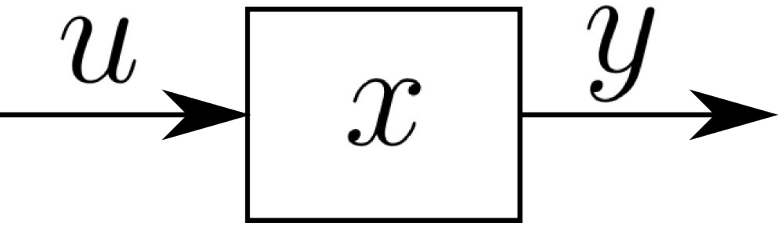
- Discrete-time model:

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k$$

- Cost minimization:

$$J_t = \frac{1}{2}(x - x_r)' Q_t (x - x_r) + \frac{1}{2}(u - u_r)' R_t (u - u_r)$$

Linear dynamic system, optimum



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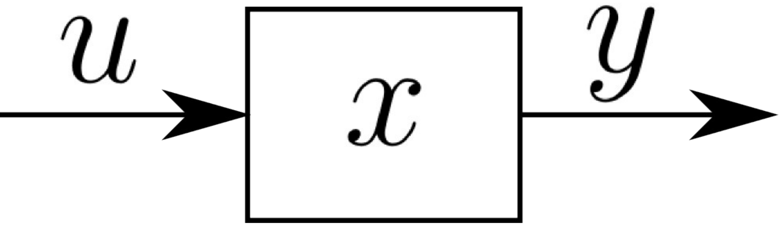
- Constraints:

$$u \in \mathcal{U} \qquad u_{min} \leq u \leq u_{max}$$
$$x \in \mathcal{X} \qquad x_{min} \leq x \leq x_{max}$$
$$y \in \mathcal{Y} \qquad y_{min} \leq y \leq y_{max}$$

Linear dynamic system, optimum



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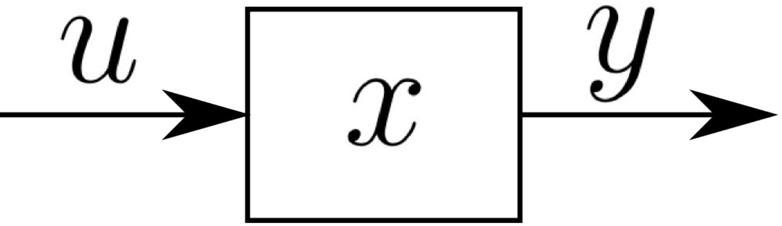
- Constraints:

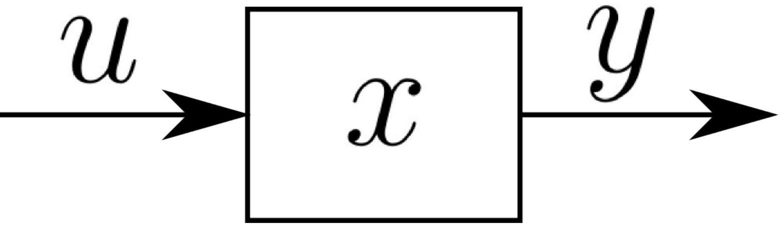
$$u_{min} \leq Uu \leq u_{max}$$
$$x_{min} \leq Xx \leq x_{max}$$
$$y_{min} \leq Yy \leq y_{max}$$

Planning the future



$$J = \sum_{k=0}^{N-1} \frac{1}{2} x'_k Q x_k + g'_x x_k + \frac{1}{2} u'_k R u_k + g'_u u_k$$

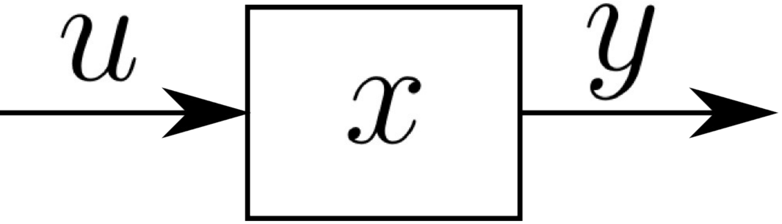
Planning the future



$$J = \sum_{k=0}^{N-1} \frac{1}{2} x_k' Q x_k + g_x' x_k + \frac{1}{2} u_k' R u_k + g_u' u_k$$

$$u^* = \arg\min_u J$$

Planning the future



$$J = \sum_{k=0}^{N-1} \frac{1}{2} x_k' Q x_k + g_x' x_k + \frac{1}{2} u_k' R u_k + g_u' u_k$$

$$u^* = \arg\min_u J \qquad \text{s.t.} \qquad \begin{aligned} x_{k+1} &= A x_k + B u_k \\ x_{k+1} &\in \mathcal{X} \\ u_k &\in \mathcal{U} \end{aligned}$$

Quadratic program

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$J = z' H z + g' z$$

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

Quadratic program

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$J = z' H z + g' z$$

Hessian

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

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$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

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Hessian

Gradient

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

Quadratic program

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Hessian

Gradient

Affine set

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

Quadratic program

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Hessian

Gradient

Affine set

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

Polyhedron

Quadratic program

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Polyhedron

Dual method

Solving the QP

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- Explicit MPC: solving the problem in advance

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  - Solution piecewise linear

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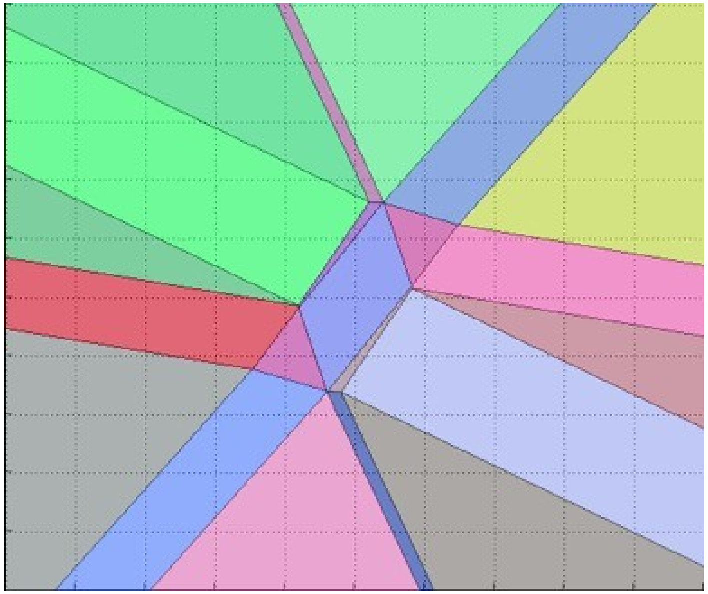
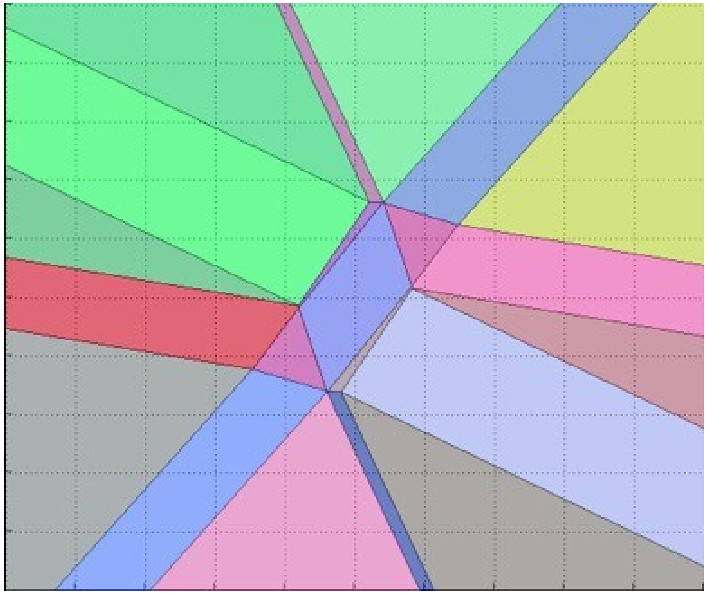


(http://www.seas.upenn.edu/~ese680/papers/explicit_linear_mpc.pdf)

Solving the QP

- Explicit MPC: solving the problem in advance
  - Solution piecewise linear
  - Active and inactive constraints:



$$a \leq b$$

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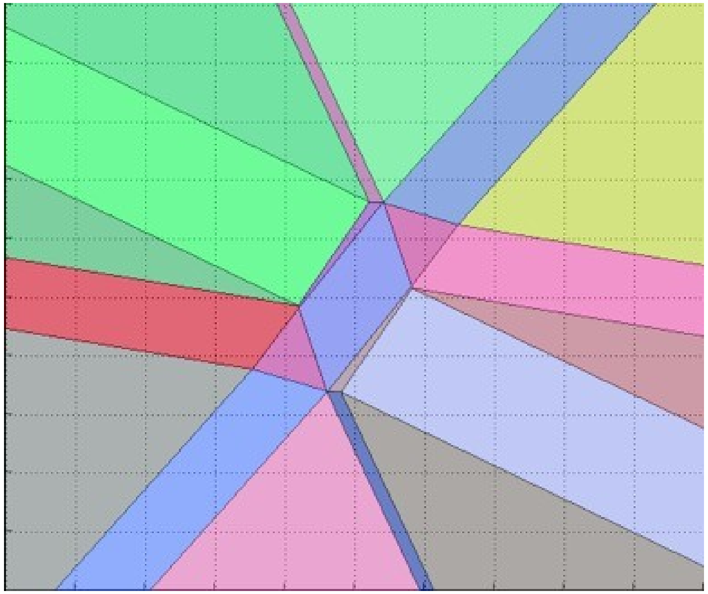

$$a \leq b$$

$$a < b$$

(http://www.seas.upenn.edu/~ese680/papers/explicit_linear_mpc.pdf)

Solving the QP

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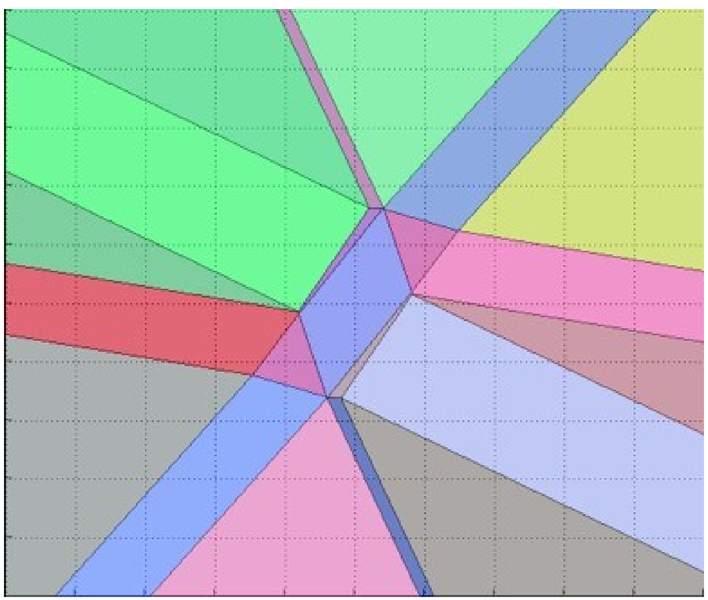

$$a \leq b$$

$$a < b \qquad a = b$$

(http://www.seas.upenn.edu/~ese680/papers/explicit_linear_mpc.pdf)

Solving the QP

- Explicit MPC: solving the problem in advance
  - Solution piecewise linear
  - Active and inactive constraints:

$$a \leq b$$

$$a < b \qquad a = b$$

- Active set methods

Solving the QP

- Explicit MPC: solving the problem in advance
  - Solution piecewise linear
  - Active and inactive constraints:

$$
\begin{tikzcd}[column sep=small]
 & a \leq b \arrow[dl] \arrow[dr] & \\
a < b & & a = b
\end{tikzcd}
$$

- Active set methods
  - Online search for active constraints

Solving the QP

- Explicit MPC: solving the problem in advance
  - Solution piecewise linear
  - Active and inactive constraints:

$$a \leq b$$

$$a < b \qquad a = b$$

- Active set methods
  - Online search for active constraints
  - Iterative

Solving the QP

- Interior point methods

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  - Cost is modified with a steep smooth function close to polyhedral constraints

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- First order methods

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- Interior point methods
  - Cost is modified with a steep smooth function close to polyhedral constraints
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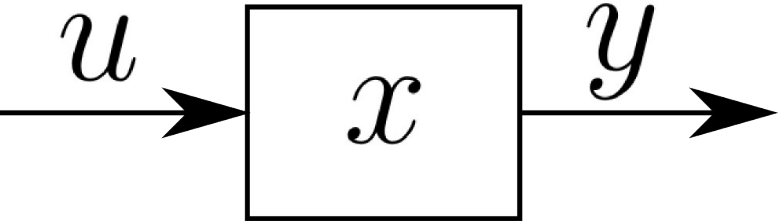
- First order methods
  - Inspired by gradient method

Solving the QP

- Interior point methods
  - Cost is modified with a steep smooth function close to polyhedral constraints
  - Modified problem solved with unconstrained optimization method
  - Modification decreased iteratively
- First order methods
  - Inspired by gradient method
  - Simple, low order of convergence

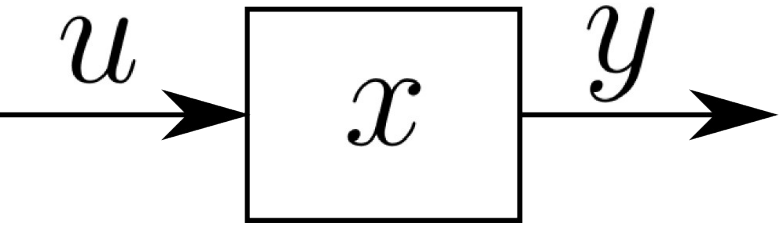
Existence of a solution



$$J = \sum_{k=0}^{N-1} \frac{1}{2} x_k' Q x_k + g_x' x_k + \frac{1}{2} u_k' R u_k + g_u' u_k$$

$$u^* = \arg\min_u J \quad \text{s.t.} \quad \begin{aligned} x_{k+1} &= A x_k + B u_k \\ x_{k+1} &\in \mathcal{X} \\ u_k &\in \mathcal{U} \end{aligned}$$

Existence of a solution



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Existence of a solution

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$J = z' H z + g' z$$

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

Existence of a solution

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

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$$z \in \mathcal{Z}$$

Soft constraints

Making the QP convenient

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad z = \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ x_{N-1} \\ u_{N-1} \end{bmatrix} \quad z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$z' = P \cdot z$$

Simplifying the QP

Move blocking, constraint reduction

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$J = z' H z + g' z$$

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

Move blocking, constraint reduction

$$J = z'Hz + g'z$$

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

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Move blocking, constraint reduction

$$J = z' H z + g' z$$

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

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$$z \in \mathcal{Z}$$

$$J = \sum_{k=0}^{N-1} \frac{1}{2} x_k' Q x_k + g_x' x_k + \frac{1}{2} u_k' R u_k + g_u' u_k$$

Move blocking, constraint reduction

$$J = z'Hz + g'z$$

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Move blocking, constraint reduction

$$J = z'Hz + g'z$$

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Move blocking, constraint reduction

$$J = z'Hz + g'z$$

$$A_c z = b_c$$

$$z \in \mathcal{Z}$$

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad x_{k+1} = Ax_k + Bu_k$$

$$x_{k+1} \in \mathcal{X}$$

$$u_k \in \mathcal{U}$$

$$J = \sum_{k=0}^{N-1} \frac{1}{2} x_k' Q x_k + g_x' x_k + \frac{1}{2} u_k' R u_k + g_u' u_k$$

Cat

(http://animalswalls.blogspot.si/2011/08/cat-wallpapers.html)

Gradient method (ordinary)

(F. Borrelli, A. Bemporad, M. Morari, Predictive Control for linear and hybrid systems, 2015.)

unconstrained

Algorithm 4.1 Gradient method for smooth convex optimization
Input: Initial iterate $z^0 \in \mathbb{R}^s$, Lipschitz constant L of ∇f
Output: Approximate minimizer

repeat
$z^{k+1} = z^k - \frac{1}{L} \nabla f(z^k)$
until stopping criterion is satisfied

constrained

Algorithm 4.6 Gradient method for constrained smooth convex optimization
Input: Initial iterate $z^0 \in S$, Lipschitz constant L of ∇f
Output: Approximate minimizer

repeat
$z^{k+1} = \pi_S \left(z^k - \frac{1}{L} \nabla f(z^k) \right)$
until stopping criterion is satisfied

Lipschitz constant

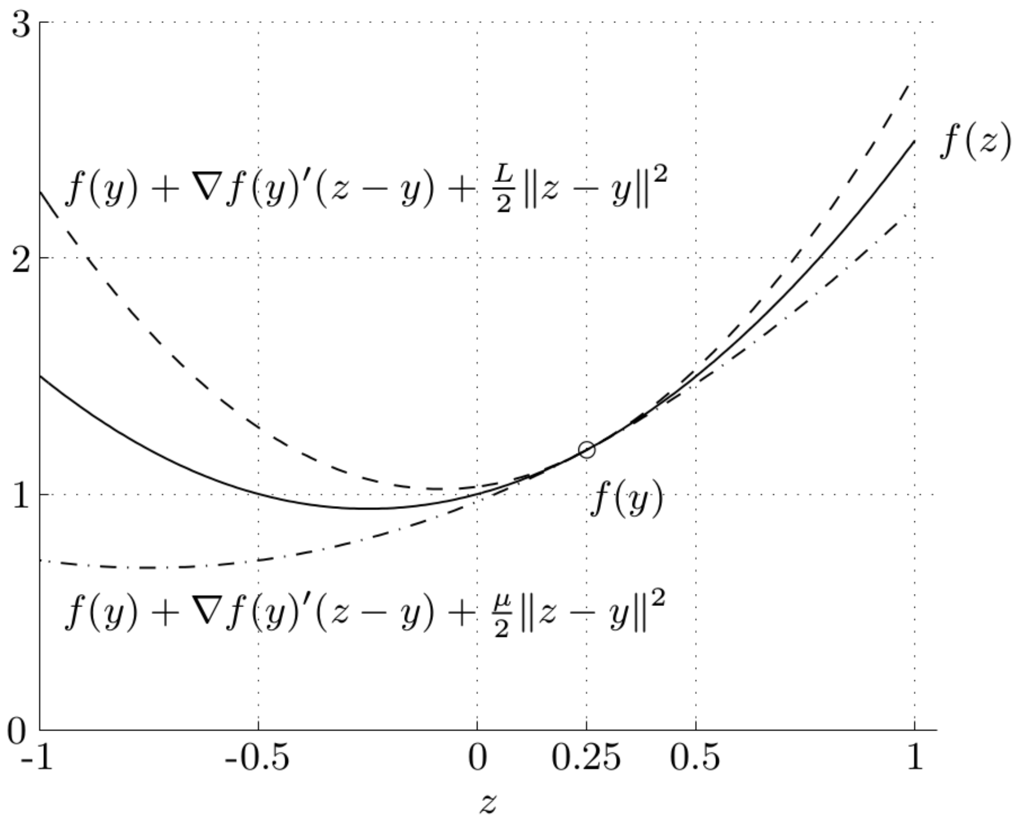

Figure 4.1 Upper and lower bounds on functions: The quadratic upper bound stems from L-smoothness of f (Definition 4.2), while the quadratic lower bound is obtained from strong convexity of f (Theorem 4.1). In this example, $f(z) = z^2 + \frac{1}{2}z + 1$ and we have chosen $L = 3$ and $\mu = 1$ for illustration.

Gradient method (ordinary)

(F. Borrelli, A. Bemporad, M. Morari, Predictive Control for linear and hybrid systems, 2015.)

unconstrained

Algorithm 4.1 Gradient method for smooth convex optimization
Input: Initial iterate $z^0 \in \mathbb{R}^s$, Lipschitz constant L of ∇f
Output: Approximate minimizer

repeat
$z^{k+1} = z^k - \frac{1}{L}\nabla f(z^k)$
until stopping criterion is satisfied

constrained

Algorithm 4.6 Gradient method for constrained smooth convex optimization
Input: Initial iterate $z^0 \in S$, Lipschitz constant L of ∇f
Output: Approximate minimizer

repeat
$z^{k+1} = \pi_S\left(z^k - \frac{1}{L}\nabla f(z^k)\right)$
until stopping criterion is satisfied

Gradient method (fast)

(F. Borrelli, A. Bemporad, M. Morari, Predictive Control for linear and hybrid systems, 2015.)

unconstrained

Algorithm 4.2 Fast gradient method for smooth convex optimization
Input: Initial iterates $z^0 \in \mathbb{R}^s$, $y^0 = z^0$; $\alpha^0 = \frac{1}{2}(\sqrt{5}-1)$, Lipschitz constant L of ∇f
Output: Approximate minimizer

repeat

$$z^{k+1} = y^k - \tfrac{1}{L}\nabla f(y^k)$$
$$\alpha^{k+1} = \frac{\alpha^k}{2}\left(\sqrt{\alpha^{k^2}+4} - \alpha^k\right)$$
$$\beta^k = \frac{\alpha^k(1-\alpha^k)}{\alpha^{k^2}+\alpha^{k+1}}$$
$$y^{k+1} = z^{k+1} + \beta^k(z^{k+1} - z^k)$$

until stopping criterion is satisfied

constrained

Algorithm 4.7 Fast gradient method for constrained smooth strongly convex optimization
Input: Initial iterates $z^0 \in S$, $y^0 = z^0$; $0 < \sqrt{\mu/L} \leq \alpha^0 < 1$, Lipschitz constant L of ∇f, strong convexity parameter μ of f
Output: Approximate minimizer

repeat

$$z^{k+1} = \pi_S\left(y^k - \tfrac{1}{L}\nabla f(y^k)\right)$$
Compute $\alpha^{k+1} \in (0,1)$: $\alpha^{{k+1}^2} = (1-\alpha^{k+1})\alpha^{k^2} + \frac{\mu\alpha^{k+1}}{L}$
$$\beta^k = \frac{\alpha^k(1-\alpha^k)}{\alpha^{k^2}+\alpha^{k+1}}$$
$$y^{k+1} = z^{k+1} + \beta^k(z^{k+1} - z^k)$$

until stopping criterion is satisfied

Lagrange multipliers

("Lagrange multiplier." Wikipedia, The Free Encyclopedia, 2015.)

$$\max_x f(x) \quad \text{s.t.} \quad g(x) = 0$$

Lagrange multipliers

("Lagrange multiplier." Wikipedia, The Free Encyclopedia, 2015.)

$$\max_{x} f(x) \quad \text{s.t.} \quad g(x) = 0$$

$$\mathcal{L}(x, \lambda) \coloneqq f(x) + \lambda \cdot g(x)$$

Lagrange multipliers

("Lagrange multiplier." Wikipedia, The Free Encyclopedia, 2015.)

$$\max_{x} f(x) \quad \text{s.t.} \quad g(x) = 0$$

$$\mathcal{L}(x,\lambda) \coloneqq f(x) + \lambda \cdot g(x)$$

$$\nabla_{x,\lambda}\mathcal{L}(x,\lambda) = 0$$

Lagrange and dual functions

(Stephen Boyd & Lieven Vandenberghe, Convex Optimization, Cambridge University Press 2004)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

Lagrange and dual functions

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x),$$

Lagrange and dual functions

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

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$$g(\lambda, \nu) \leq p^\star \qquad \lambda \succeq 0$$

Lagrange and dual functions

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$$g(\lambda, \nu) \le p^\star \qquad \lambda \succeq 0$$

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

Lagrange and dual functions

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

$$g(\lambda, \nu) \leq p^\star \qquad \lambda \succeq 0 \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

We mentioned at the beginning of §5.5.3 that if strong duality holds and a dual optimal solution $(\lambda^\star, \nu^\star)$ exists, then any primal optimal point is also a minimizer of $L(x, \lambda^\star, \nu^\star)$. This fact sometimes allows us to compute a primal optimal solution from a dual optimal solution.

FGMdual

(generalized fast dual gradient method)

Improving Fast Dual Ascent for MPC - Part II: The Embedded Case ⋆

Pontus Giselsson *

* *Electrical Engineering, Stanford University (e-mail: pontusg@stanford.edu).*

(generalized fast dual gradient method)

Improving Fast Dual Ascent for MPC - Part II: The Embedded Case ⋆

Pontus Giselsson *

* *Electrical Engineering, Stanford University (e-mail: pontusg@stanford.edu).*

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2} y^T H y \\
\text{subject to} & Ay = b\bar{x} \\
 & By = v \\
 & \underline{d} \leq v \leq \bar{d}
\end{array}
$$

Letting $H_A = AH^{-1}A^T$, the algorithm becomes

$$y^k = H^{-1}(A^T H_A^{-1}(AH^{-1}B^T v^k + b\bar{x}) - B^T v^k) \tag{47}$$

$$\mu^k = \text{prox}_{g^\star}^{\mathbf{L}_\mu}(v^k + \mathbf{L}_\mu^{-1} B y^k) \tag{48}$$

$$t^{k+1} = \frac{1+\sqrt{1+4(t^k)^2}}{2} \tag{49}$$

$$v^{k+1} = \mu^k + \left(\frac{t^k - 1}{t^{k+1}}\right)(\mu^k - \mu^{k-1}) \tag{50}$$

FGMdual

(generalized fast dual gradient method)

Improving Fast Dual Ascent for MPC - Part II: The Embedded Case ⋆

Pontus Giselsson *

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$$\begin{array}{ll} \text{minimize} & \frac{1}{2} y^T H y \\ \text{subject to} & Ay = b\bar{x} \\ & By = v \\ & \underline{d} \leq v \leq \bar{d} \end{array}$$

Letting $H_A = AH^{-1}A^T$, the algorithm becomes

$$y^k = H^{-1}(A^T H_A^{-1}(AH^{-1}B^T v^k + b\bar{x}) - B^T v^k) \quad (47)$$

$$\mu^k = \mathrm{prox}_{g^\star}^{\mathbf{L}_\mu}(v^k + \mathbf{L}_\mu^{-1} B y^k) \quad (48)$$

$$t^{k+1} = \frac{1+\sqrt{1+4(t^k)^2}}{2} \quad (49)$$

$$v^{k+1} = \mu^k + \left(\frac{t^k - 1}{t^{k+1}}\right)(\mu^k - \mu^{k-1}) \quad (50)$$

$$\mathrm{prox}_\psi^{\mathbf{L}}(x) := \arg\min_y \left\{ \psi(y) + \frac{1}{2}\|y - x\|_{\mathbf{L}}^2 \right\}$$

$$\mathrm{prox}_{g^\star}^{\mathbf{L}}(x) + \mathbf{L}^{-1}\mathrm{prox}_g^{\mathbf{L}^{-1}}(\mathbf{L}x) = x$$

QPgen

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- Code generator:

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  - Problem described in MATLAB

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  - MATLAB code is long

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  - Problem described in MATLAB
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  - FGMdual one of the supported algorithms
- Knowing the code:
  - MATLAB code is long
  - Generated code is confusing

```c
while ((jj < *max_iter) && (cond < 0)) {

jj++;

copy_vec_part_negate(v,tmp_var_p,40);

mat_vec_mult_sparse(&CT,tmp_var_p,tmp_var_n);

vec_sub(tmp_var_n,q1,tmp_var_n,60);

stack_vec(tmp_var_n,q2,rhs,60,40);

perm_fwdsolve(&L,p,rhs,tmp_var_nm);

mat_vec_mult_sparse(&Dinv,tmp_var_nm,tmp_var_nm2);

backsolve_perm(&LT,p,tmp_var_nm2,tmp_var_nm);

copy_vec_part(tmp_var_nm,x,60);

mat_vec_mult_sparse(&C,x,tmp_var_p);

vec_add(v,tmp_var_p,tmp_var_p,40);

copy_vec_part(tmp_var_p,arg_prox_h,40);

clip_soft(tmp_var_p,l,u,(double *) &soft,40);

mat_vec_mult_diag(&Einv,tmp_var_p,y);

copy_vec_part(lambda,lambda_old,40);

vec_sub(arg_prox_h,tmp_var_p,lambda,40);

vec_sub(lambda,lambda_old,tmp_var_p,40);

theta_old = theta;

theta = (1+sqrt(1+4*pow(theta_old,2)))/2;

scalar_mult((theta_old-1)/theta,tmp_var_p,40);

copy_vec_part(v,v_old,40);

vec_add(tmp_var_p,lambda,v,40);

if (mod(jj,10) == 0) {
 cond = check_stop_cond_FGM(&Einv,lambda,lambda_old,tmp_var_p,tm
 }

restart(lambda,lambda_old,v,v_old,tmp_var_p,tmp_var_p2,40);

}
```

ponavljaj

$$tmp\_var\_p = -v^k$$

$$tmp\_var\_n = \mathsf{B}^{\mathrm{T}} tmp\_var\_p \left(= -\mathsf{B}^{\mathrm{T}} v^k\right)$$

$$tmp\_var\_n = tmp\_var\_n - g \left(= -\mathsf{B}^{\mathrm{T}} v^k - g\right)$$

$$rhs = \begin{bmatrix} tmp\_var\_n \\ b\bar{x} \end{bmatrix} \left(= \begin{bmatrix} -\mathsf{B}^{\mathrm{T}} v^k - g \\ b\bar{x} \end{bmatrix}\right)$$

$$\mathsf{L} tmp\_var\_nm = rhs, \quad tmp\_var\_nm = \mathsf{L}^{-1} rhs \qquad \left(\mathsf{K} = \mathsf{L}\mathsf{D}\mathsf{L}^{\mathrm{T}}\right)$$

$$tmp\_var\_nm2 = \mathsf{D}^{-1} tmp\_var\_nm \left(= \mathsf{D}^{-1}\mathsf{L}^{-1} rhs\right)$$

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$$\begin{bmatrix} y \\ \xi \end{bmatrix} = tmp\_var\_nm$$

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$$tmp\_var\_p = max\left(min\left(tmp\_var\_p, u\right), l\right)$$

$$\mu^{k-1} = \mu^k$$

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dokler pogoji niso izpolnjeni

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$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} y^k \\ \xi \end{bmatrix} = \begin{bmatrix} -B^T v^k \\ b\bar{x} \end{bmatrix}$$

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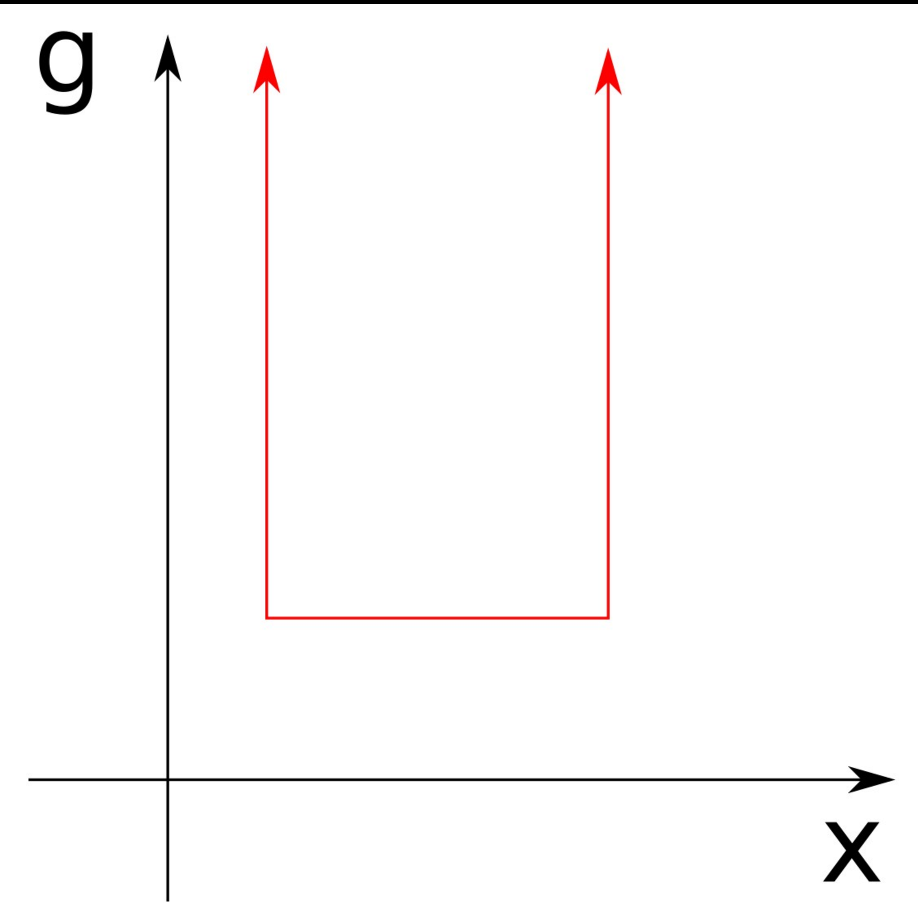

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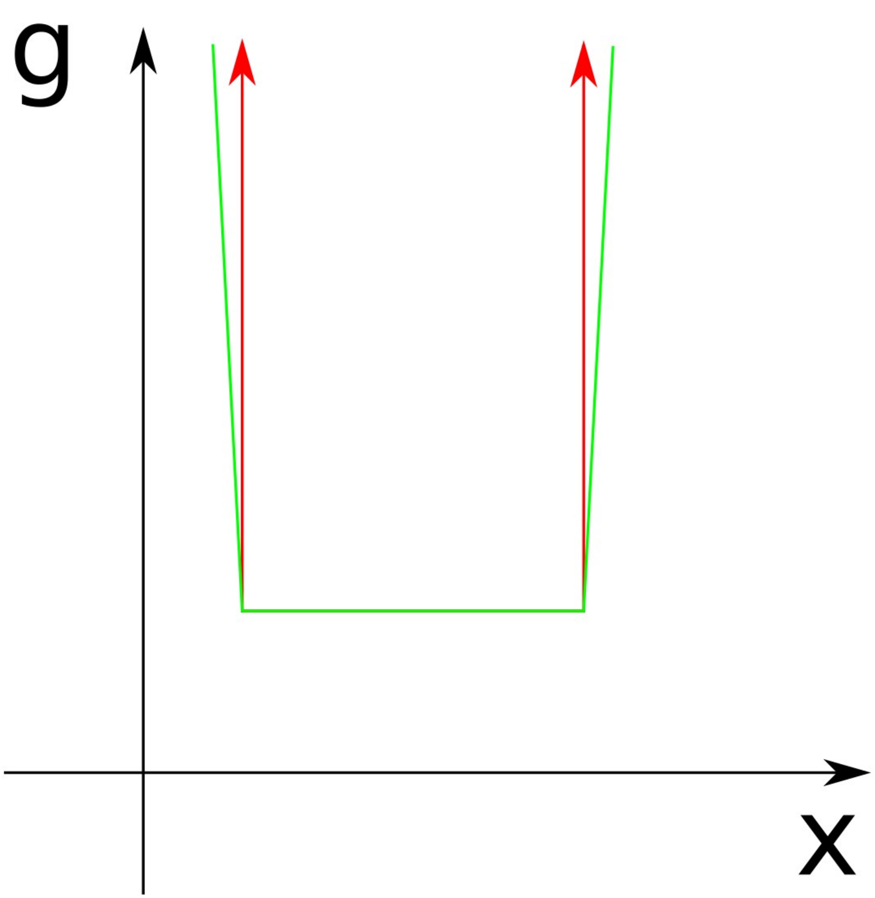

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 mat_vec_mult_sparse(&C,x,tmp_var_p);
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 copy_vec_part(tmp_var_p,arg_prox_h,40);
 clip_soft(tmp_var_p,l,u,(double *) &soft,40);
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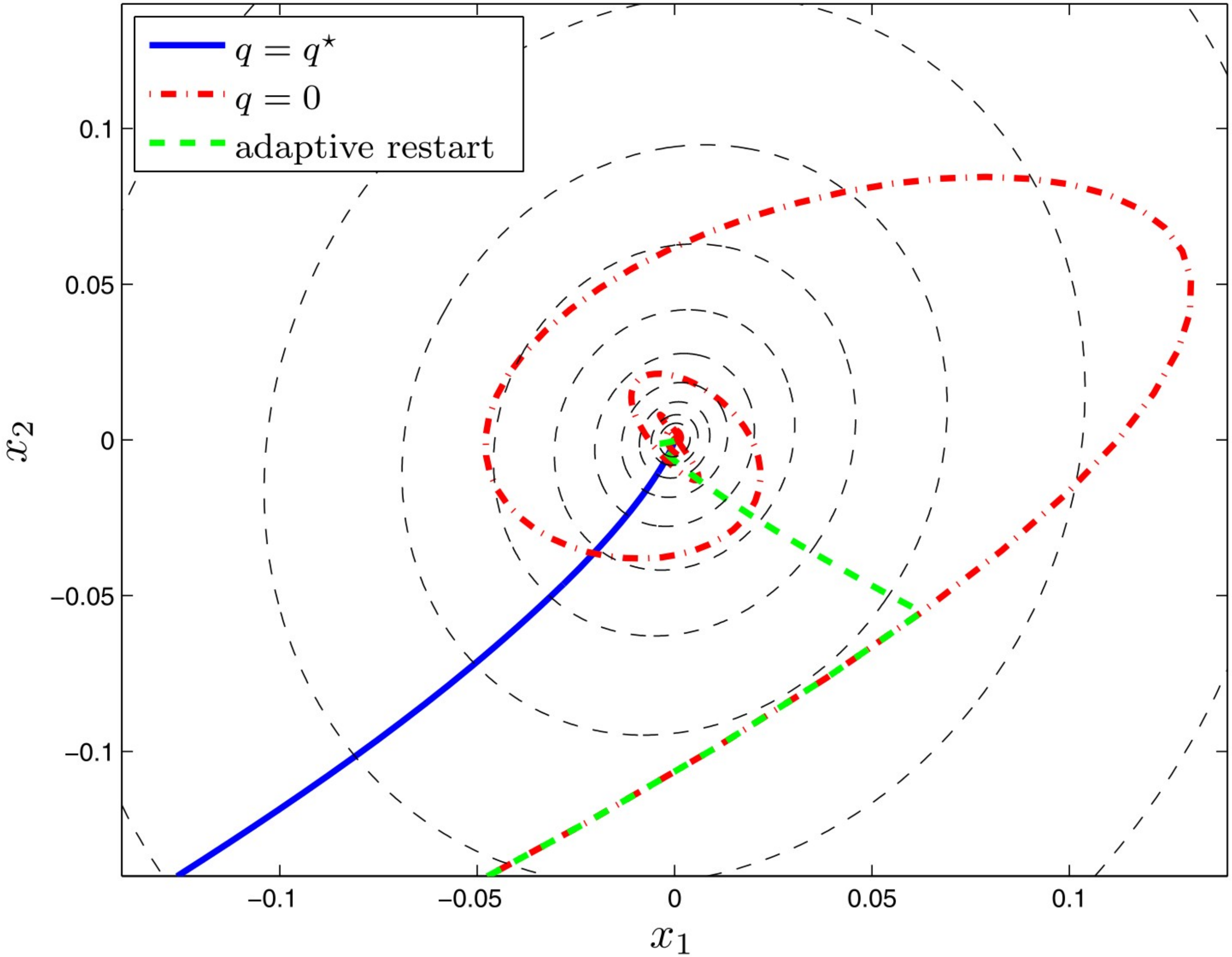
 copy_vec_part(lambda,lambda_old,40);
 vec_sub(arg_prox_h,tmp_var_p,lambda,40);
 vec_sub(lambda,lambda_old,tmp_var_p,40);
 theta_old = theta;
 theta = (1+sqrt(1+4*pow(theta_old,2)))/2;
 scalar_mult((theta_old-1)/theta,tmp_var_p,40);
 copy_vec_part(v,v_old,40);
 vec_add(tmp_var_p,lambda,v,40);
 if (mod(jj,10) == 0) {
  cond = check_stop_cond_FGM(&Einv,lambda,lambda_old,tmp_var_p,tmp_var_p2,40,1e-09);
 }
 restart(lambda,lambda_old,v,v_old,tmp_var_p,tmp_var_p2,40);
}
```

```c
static void restart(double *x, double *x_old, double *y, double *y_old, double *tmp_v
 double test;
 vec_sub(y_old,x,tmp_var_p,n);
 vec_sub(x,x_old,tmp_var_p2,n);
 test = scalar_prod(tmp_var_p,tmp_var_p2,n);
 if (test > 0) {
 copy_vec_part(x_old,y,n);
 copy_vec_part(x_old,x,n);
 }
 }
```

Restart

(B. O'Donoghue, E. Candès, Adaptive Restart for Accelerated Gradient Schemes, Found Comput Math 2013.)



Complexity certification

(S. Richter, Computational Complexity Certification of Gradient Methods for Real-Time Model Predictive Control, ETH Zurich 2012)

- Upper bound on the required number of floating point operations that in turn stems from an upper bound on the iteration count

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Complexity certification

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- Active set:
  - Finite termination
  - Worst case number of iterations huge
  - Early termination unexplored
- Interior point:
  - Certificates very conservative
- First order:
  - Certificates within few orders of magnitude

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  - Derived for certain (simpler) algorithms

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Nesterov

(Y. Nesterov, Introductory lectures on convex optimization : a basic course, Springer 2004)

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- MPC with some QP
- Duality: partial Lagrange relaxation (simple state constraints), shows that complete is worse, but complete is what we need
- Duality: dual problem not strongly concave

Temporary conclusion

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- Restarting?